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NUMERICAL SOLUTION OF COMBINED CONVECTIVE HEAT TRANSFER OF MICROPOLAR FLUID IN AN ANNULUS OF TWO VERTICAL PIPES

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Abstract—The problem of free and forced convective heat transfer of micropolar fluid in an annulus of two vertical pipes has been studied by applying Runge–Kutta–Merson method. It has been found that, when the temperature of the pipes decreases with height, the presence of micropolar elements delay the onset of instability. On the other hand, when the temperature increases with height it has been noticed that microrotation enhances the back flow. Effect of micropolar parameters and Rayleigh number on heat transfer are also discussed in detail.

NOMENCLATURE

a, b, radii of inner and outer cylinders;
r, ϕ, z, cylindrical polar coordinates;
τ/b, uniform temperature gradient per unit length along z-axis;
T_0, T_1, temperatures of the outer and inner pipes at $z = 0$;
T, temperature of the fluid;
P, pressure;
ν, kinematic viscosity;
ρ, density of the fluid;
ρ_b, density of the fluid at $z = 0$ and $r = b$;
λ, thermal diffusivity;
g, acceleration due to gravity;
μ, κ, γ, material constants;
W, z-component of velocity;
Ω, ϕ-component of microrotation;
β, coefficient of volume expansion;
Ra, Rayleigh number;
S, non-dimensional parameter;
R, A, non-dimensional micropolar parameters;
η, ζ, non-dimensional co-ordinates;
s, ratio of the radii.

1. INTRODUCTION

IN RECENT years, the study of microcontinuum fluid mechanics has remarkable progress. Several theories describing fluids consisting of molecules, whose lengths are not negligible when compared with the characteristic length of the geometry, have been formulated. Depending upon the approach of formulation fluids are called by various names such as—simple microfluids, micropolar fluids, deformable directed fluids, polar fluids, dipolar fluids, anisotropic fluids etc.

The theory of micropolar fluids developed by Eringen [1] deals with viscous fluids in which the microconstituents are rigid and spherical or randomly oriented. This theory is supposed to describe polymeric fluids, liquid crystals, animal blood etc. The recent literature is replete with a wide variety of problems studied by various authors in this field. The application of this theory to biomechanics has been an exciting topic of current interest. Ariman [2], Turk *et al.* [3, 4] have analysed some steady and unsteady blood flows and have seen good agreement with the experimental results. Kazakai and Ariman [5] have introduced the theory of heat conducting micropolar fluids and have analysed the flows between two parallel plates. Ariman [6] has investigated heat conduction in blood. Balaram and Sastry [7] have discussed the free convective heat transfer in a parallel plate vertical channel and have shown that the fluid acts as a coolant.

Our interest in this paper is to examine the combined effect of buoyancy force and the pressure gradient on the flow and heat transfer of a micropolar fluid in between two concentric cylinders. In Newtonian fluid, Morton [8] has investigated the laminar convection in a vertical pipe and has shown that the fluid is more stable in the case when the pipe temperature increases with height than in the case when it decreases. In a recent paper Gupta [9] has extended the analysis of Morton to unsteady case and has examined the phenomenon of convective heat transfer in a vertical pipe, and in coaxial pipes of circular and elliptic cross-sections.

As the problem on hand is governed by the boundary value problem consisting of three simultaneous second order linear differential equations, it appears that it is not possible to obtain an analytical solution. Hence, we adopt the Runge–Kutta–Merson method [10], which automatically adjusts the step length, and obtain the numerical solution.

2. FORMULATION AND SOLUTION OF THE PROBLEM

We consider the steady laminar flow of an incompressible micropolar fluid between two vertical coaxial circular cylinders of radii a and b ($b > a$). Let z-axis be coinciding with the axis of the cylinders. We take the temperature of the inner and outer cylinders as $T_a = T_1 + \tau z/b$ and $T_b = T_0 + \tau z/b$ respectively, and the temperature of the fluid as $T = T_b - \theta(r)$.

Under Boussinesq approximation, the equations of motion and energy governing the flow are given by

$$(\mu+\kappa)\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}\right)W+\frac{\kappa}{r}\frac{d}{dr}(r\Omega)-\frac{\partial p}{\partial z}-\rho g = 0, \quad (2.1)$$

$$\gamma\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{1}{r^2}\right)\Omega-\kappa\frac{dW}{dr}-2\kappa\Omega = 0, \quad (2.2)$$

$$\lambda\left(\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}\right) = W\frac{\partial T}{\partial z}. \quad (2.3)$$

In the above equations, μ is the coefficient of viscosity, κ the coefficient of micro-viscosity and γ is the coefficient of couple stress. It can be seen that $\kappa = 0$ and $\gamma = 0$ correspond to the Newtonian case. On the other hand, if $\kappa \neq 0$ and $\gamma = 0$, there is no torque arising from micro stress, and the equation (2.2) indicates that the microrotational velocity turns out to be half of vorticity of the fluid.

Using the equation of state $\rho = \rho_b\{1+\beta(T_b-T)\}$, the equation (2.1) can be rewritten as

$$(\mu+\kappa)\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}\right)W+\rho_b\beta g(T-T_b) -\left(\frac{\partial p}{\partial z}+\rho_b g\right)+\frac{\kappa}{r}\frac{d}{dr}(r\Omega) = 0. \quad (2.4)$$

Introducing the non-dimensional variables

$$\eta = \frac{r}{b},\quad \zeta = \frac{z}{b},\quad w = \frac{bW}{\lambda},\quad \omega = \frac{\Omega b^2}{\lambda},\quad \Theta = \frac{\theta}{\tau},$$

into (2.2)–(2.4) we get

$$(1+R)(D^2+\eta^{-1}D)w+R(D+\eta^{-1})\omega = -S+Ra\Theta, \quad (2.5)$$

$$A(D^2+\eta^{-1}D-\eta^{-2})\omega-RDw-2R\omega = 0, \quad (2.6)$$

$$(D^2+\eta^{-1}D)\Theta+w = 0, \quad (2.7)$$

where

$$D = \frac{d}{d\eta},\qquad Ra = \frac{b^3\beta g\tau}{\lambda\nu},$$

$$S = -\frac{b^3}{\lambda\nu}\left(\frac{1}{\rho_b}\frac{\partial p}{\partial z}+g\right),$$

$$R = \frac{\kappa}{\mu}\quad\text{and}\quad A = \frac{\gamma}{\mu b^2}.$$

Here Ra positive (negative) indicates the case when the pipe temperature increases (decreases) with height. Physically this means, that the ascending (descending) cool fluid is heated or the descending (ascending) hot fluid is cooled steadily.

The boundary conditions appropriate for the problem are:

$$\eta = s:\ w = 0,\ \omega = 0,\ \Theta = \frac{T_0-T_1}{\tau} = Tr, \quad (2.8)$$

$$\eta = 1:\ w = 0,\ \omega = 0,\ \Theta = 0, \quad (2.9)$$

in which $s = a/b$.

The equations (2.5)–(2.7) together with the boundary conditions (2.8) and (2.9) constitute a boundary value problem.

To solve the problem by Runge–Kutta–Merson method, we convert the above system of equations into six first order equations. Considering the homogeneous equations (which can be obtained by suppressing the right hand side function if any) along with the following sets of conditions

$$w = 0,\ Dw = 1,\ \omega = 0,\ D\omega = 0,\ \Theta = 0,\ D\Theta = 0;$$

$$w = 0,\ Dw = 0,\ \omega = 0,\ D\omega = 1,\ \Theta = 0,\ D\Theta = 0;$$

$$w = 0,\ Dw = 0,\ \omega = 0,\ D\omega = 0,\ \Theta = 0,\ D\Theta = 1;$$

we obtain three particular solutions w_i, ω_i, and $\Theta_i (i = 1, 2, 3)$ by forward integration. Similarly solving the nonhomogeneous equations subject to the conditions

$$w = 0,\ \omega = 0,\ \Theta = Tr,\ Dw = 0,\ D\omega = 0,\ D\Theta = 0,$$

we get the particular integrals w_0, ω_0 and Θ_0. Thus the solution can be written as

$$w = A_i w_i + w_0,$$

$$\omega = A_i\omega_i+\omega_0,$$

$$\Theta = A_i\Theta_i+\Theta_0,$$

where A_i are the arbitrary constants which are to be determined by applying the boundary conditions at $\eta = 1$.

3. DISCUSSION OF THE RESULTS

The Fortran program written for the solution of simultaneous linear differential equations by employing Runge–Kutta–Merson method has been seen [11] to yield solution having agreement up to fourth decimal place with the exact solution. In solving the present problem, we have considered the cases: when the cylinders are (i) at unequal temperature ($Tr \neq 0$) and (ii) at equal temperature ($Tr = 0$) separately. We have fixed up the value of s as 0·5 throughout the computation.

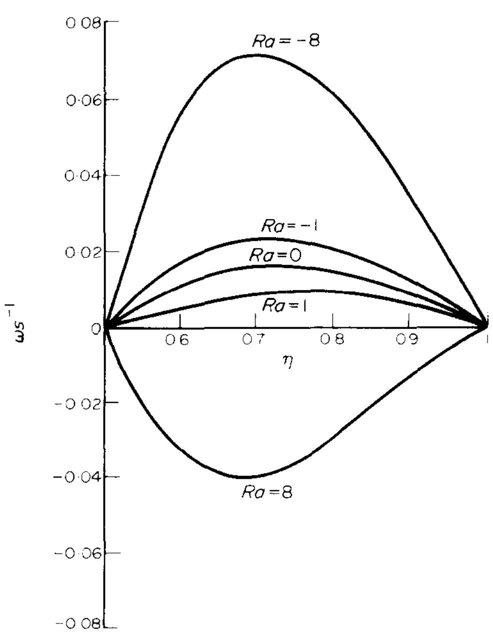

FIG. 1. Velocity distribution for different values of Ra when $R = 1$, $A = 1$, $Tr = 1$.

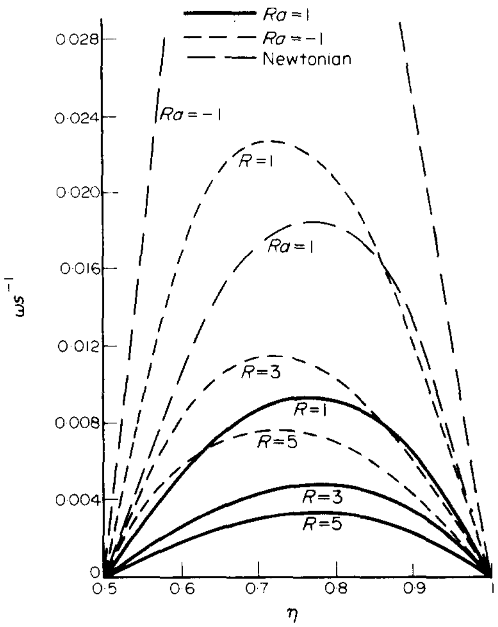

FIG. 2. Velocity distribution for different values of R and Ra for $A = 1$, $Tr = 1$.

The velocity profiles presented in Figs. 1 and 2 reveal the following interesting results. When $Ra < 0$, that is, when the temperature of the pipes decreases with height ($\tau < 0$), the buoyancy force strengthens the pressure gradient and thereby the fluid particles move with higher velocity as Ra increases. As $Ra(<0)$ assumes still higher values, in the Newtonian case, it can be seen that the fluid rises up with very high velocity and thus causing instability for the flow. However, in the present case, as there is considerable reduction in the velocity owing to the presence of micropolar elements, we notice that the onset of instability will be delayed. On the other hand, when $Ra > 0$, i.e. when the temperature of the pipes increases with height ($\tau > 0$), the fluid particles get decelerated as Ra increases and eventually lead to back flow. It is rather curious to notice that the micro-rotation enhances the back flow.

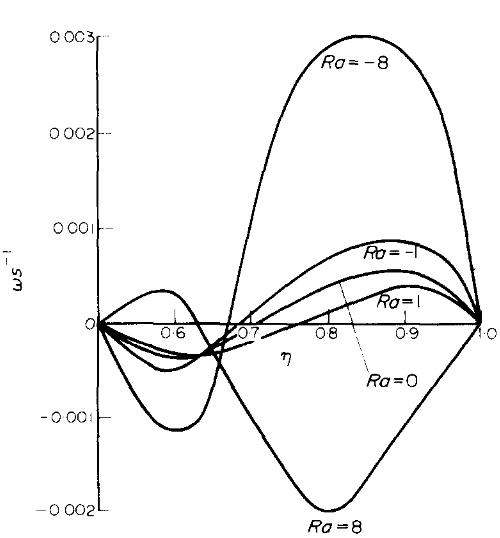

FIG. 3. Micro-rotation distribution for different values of Ra when $R = 1$, $A = 1$, $Tr = 1$.

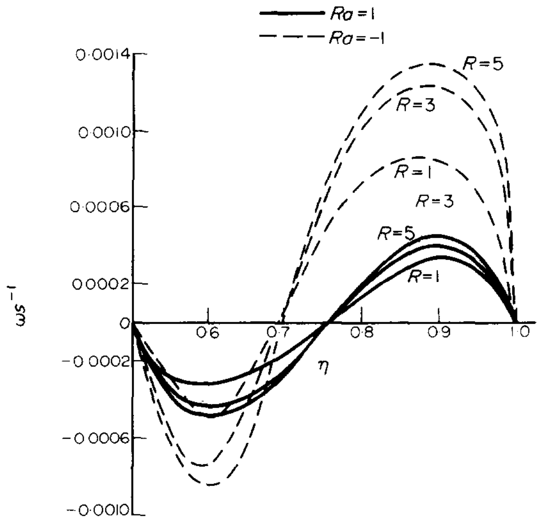

FIG. 4. Micro-rotation distribution for different values of R and Ra when $A = 1$, $Tr = 1$.

It can also be seen that the effect of natural convection is predominant at the hotter boundary for all negative values of Ra and also for those positive values of Ra for which there is no back flow.

The values of micro-rotation (Figs. 3 and 4) are found to be negligibly small at the inner boundary. As long as there is no flow reversal, the micropolar elements at the inner boundary rotate with negative angular velocity, while those at the outer boundary rotate in opposite sense. A comparison of velocity and micro-rotation profiles indicate that wherever velocity is more there the micro-rotation is less and vice versa. This implies that when fluid is advancing with higher velocity the internal microelements rotate with less angular speed.

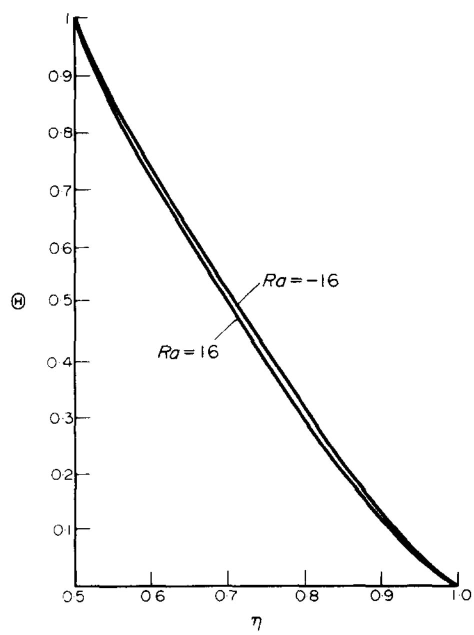

FIG. 5. Temperature distribution Θ for different values of Ra when $R = 1$, $A = 1$, $Tr = 1$.

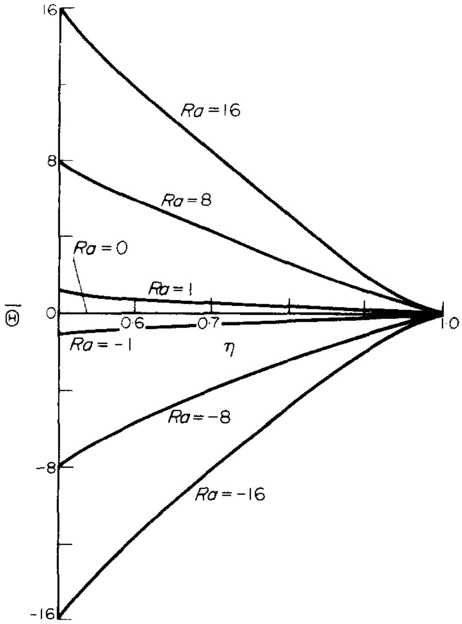

FIG. 6. Temperature distribution Θ for different values of Ra when $R = 1$, $A = 1$, $Tr = 1$.

The nondimensional temperature and the Nusselt number presented in Figs. 5 and 6, and Tables 1–3 exhibit the following fascinating results. We notice that Θ and Nu_1 are insensitive to Ra. This implies that, when τ is fixed, the material constants of the fluid, namely, thermal conductivity and kinematic viscosity do not affect the temperature of the fluid. On the other hand, by resetting the non-dimensional temperature by another variable $\bar{\Theta}$, as $\bar{\Theta} = b^3\beta g\theta/\lambda\nu = \Theta Ra$, such that all quantities except τ are fixed, we find that the effect of τ is prominent. As τ increases the temperature of the fluid increases as expected. It is seen from Tables 1 and 2 that the effect of R or Ra is less significant. It is also noticed that as R increases there is slight increase (decrease) in heat flow at the inner (outer) cylinder, while there is no effect of A. From Table 3, for all values of $Ra < 0$ and for those values of $Ra > 0$

Table 1. Nusselt number $Nu_1\left(=\frac{d\Theta}{d\eta}\right)$ for different values of R and A when $Ra = 1, -1$ and $Tr = 1$

Ra	R	A	Inner cylinder	Outer cylinder
	Newtonian		−2·8818	−1·4455
1	1	1	−2·8836	−1·4441
	3	1	−2·8845	−1·4434
	5	1	−2·8848	−1·4432
	1	3	−2·8836	−1·4441
	1	5	−2·8836	−1·4441
	Newtonian		−2·8761	−1·4491
	1	1	−2·8807	−1·4459
−1	3	1	−2·8830	−1·4443
	5	1	−2·8838	−1·4438
	1	3	−2·8807	−1·4459
	1	5	−2·8807	−1·4459

Table 2. Values of Nusselt number $Nu_1\left(=\frac{d\Theta}{d\eta}\right)$ for different values of Ra for $R = 1$, $A = 1$ and $Tr = 1$

	Inner cylinder		Outer cylinder	
Ra	Micropolar	Newtonian	Micropolar	Newtonian
−8	−2·8706	−2·8557	−1·4522	−1·4617
−1	−2·8807	−2·8761	−1·4459	−1·4491
0	−2·8822	−2·8789	−1·4450	−1·4473
1	−2·8836	−2·8818	−1·4441	−1·4455
8	−2·8937	−2·9019	−1·4379	−1·4331

Table 3. Values of Nusselt number $Nu_2\left(=\frac{d\Theta}{d\eta}\right)$ for different values of Ra for $R = 1$, $A = 1$ and $Tr = 1$

	Inner cylinder		Outer cylinder	
Ra	Micropolar	Newtonian	Micropolar	Newtonian
−8	22·9648	22·8456	11·6176	11·6936
−1	2·8807	2·8761	1·4459	1·4491
0	0	0	0	0
1	−2·8836	−2·8818	−1·4441	−1·4455
8	−23·1496	−23·2152	−11·5032	−11·4648

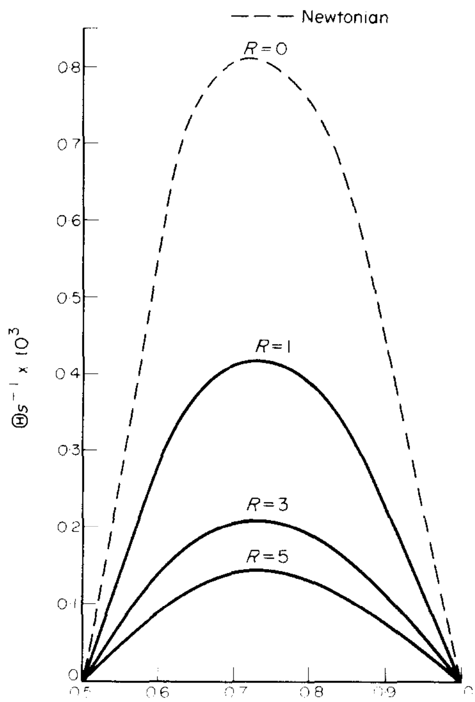

FIG. 7. Temperature profiles Θ for different values of R when $Ra = 1$, $A = 1$, $Tr = 0$.

Table 4. Nusselt number $Nu_1\left(=\frac{d\Theta}{d\eta}\right)$ for different values of R when $Ra = 1, -1$ and $Tr = 0$

Ra	R	A	Inner cylinder	Outer cylinder
	Newtonian		0·0064494	−0·0046445
	1	1	0·0032330	−0·0023277
1	3	1	0·0016290	−0·0011721
	5	1	0·0010947	−0·0007872
	1	3	0·0032282	−0·0023245
	1	5	0·0032272	−0·0023239
	Newtonian		0·0064579	−0·0046504
	1	1	0·0032351	−0·0023292
−1	3	1	0·0016296	−0·0011725
	5	1	0·001095	−0·00078745
	1	3	0·0032303	−0·0023260
	1	5	0·0032293	−0·0023254

for which there exists back flow, we find that, in micropolar case, the emission of heat is more from the boundary at higher temperature while receipt at the other boundary is less.

In the case of equal temperature we observe that there is no effect of Ra on velocity and micro-rotation. This is apparent as the values of Θ (Fig. 7) are found to be negligibly small, due to which the buoyancy force has become ineffective. We also notice from Tables 4 and 5 that the boundaries are cooled. However, this cooling effect is seen to reduce as R or Ra increases.

Table 5. Nusselt number $Nu_1\left(=\frac{d\Theta}{d\eta}\right)$ for different values of Ra when $R = 1$, $A = 1$ and $Tr = 0$

Ra	Inner cylinder Micropolar	Inner cylinder Newtonian	Outer cylinder Micropolar	Outer cylinder Newtonian
−8	0·0032425	0·00648767	−0·0023344	−0·0046714
−1	0·0032351	0·0064793	−0·0023292	−0·0046504
0	0·0032340	0·0064537	−0·0023284	−0·0046475
1	0·0032330	0·0064494	−0·0023277	−0·0046445
8	0·0032255	0·0064201	−0·0023225	−0·0046237

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SOLUTION NUMERIQUE DU TRANSFERT THERMIQUE EN CONVECTION MIXTE DE FLUIDES MICROPOLAIRES DANS UN ESPACE ANNULAIRE ENTRE DEUX TUBES VERTICAUX

Résumé—Le problème du transfert thermique par convection libre et forcée dans un fluide micropolaire dans l'espace annulaire entre deux tubes verticaux a été étudié par application de la méthode de Runge–Kutta–Merson. On a trouvé que, lorsque la température des tubes décroit avec la hauteur, la présence d'éléments micropolaires retarde la naissance d'instabilités. D'autre part, lorsque la température augmente avec la hauteur on a remarqué qu'une microrotation augmente l'écoulement de retour. Les effets des paramètres micropolaires et du nombre de Rayleigh sur le transfert de chaleur sont aussi discutés en détail.

NUMERISCHE LÖSUNG DES KOMBINIERTEN KONVEKTIVEN WÄRMEÜBERGANGS MIKROPOLARER FLUIDE IN EINEM RINGRAUM VON ZWEI SENKRECHTEN ROHREN

Zusammenfassung—Es wurde der Wärmeübergang bei freier und erzwungener Konvektion einer mikropolaren Flüssigkeit in einem Ringraum aus zwei senkrechten Rohren untersucht mit Hilfe der Runge–Kutta–Merson-Methode. Es ergab sich für Temperaturen, die mit der Rohrhöhe abnahmen, daß die Anwesenheit mikropolarer Elemente das Einsetzen von Instabilitäten verzögert. Andererseits zeigte sich, daß für Temperaturen, die mit der Rohrhöhe zunahmen, die Mikrorotation eine Rückströmung begünstigt. Der Einfluß mikropolarer Parameter und der Rayleigh–Zahl auf den Wärmeübergang wurde im Detail diskutiert.

ЧИСЛЕННОЕ РЕШЕНИЕ СОВМЕСТНОГО КОНВЕКТИВНОГО ТЕПЛООБМЕНА МИКРОПОЛЯРНОЙ ЖИДКОСТИ В КОЛЬЦЕВОМ КАНАЛЕ ДВУХ ВЕРТИКАЛЬНЫХ ТРУБОК

Аннотация — В работе исследовалась задача свободного и вынужденного конвективного теплообмена микрополярной жидкости в кольцевом канале двух вертикальных трубок методом Рунге-Кутта-Мерсона. Обнаружено, что если температура трубок падает с высотой, то наличие микрополярных элементов оттягивает возникновение неустойчивого состояния. С другой стороны, при повышении температуры трубок с высотой микровращение усиливает обратное течение. В работе подробно обсуждается влияние микрополярных параметров и числа Релея на теплообмен.